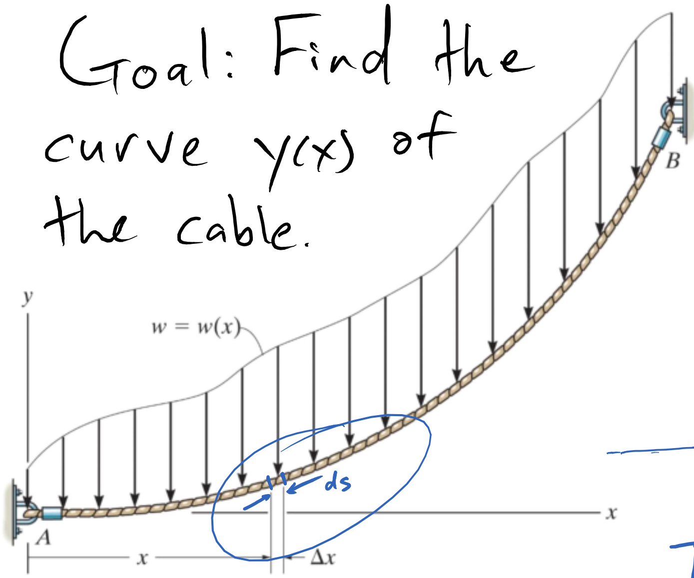
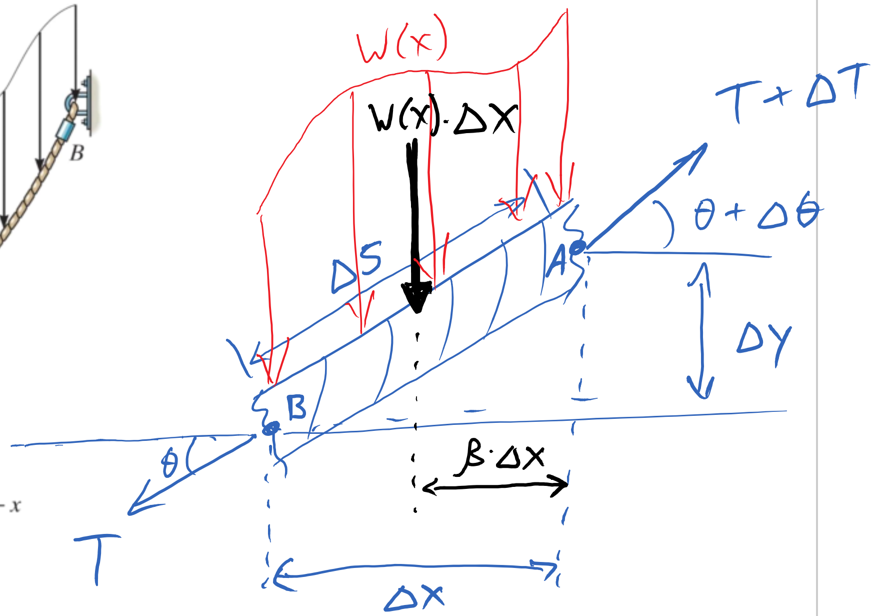
Cable subjected to a distributed load

Goal: Find the curve y(x) of the cable.





$$\Sigma F_x = 0 \Rightarrow (T+\Delta T)\cdot\cos(\theta+\Delta\theta) - T\cdot\cos\theta = 0$$

$$\Sigma F_y = 0 \Rightarrow (T+\Delta T)\cdot\sin(\theta+\Delta\theta) - T\cdot\sin\theta - W(x)\cdot\Delta x = 0$$

$$(\Sigma M)_A = 0 \Rightarrow \Delta x\cdot T\cdot\sin\theta - \Delta y\cdot T\cdot\cos\theta + W(x)\cdot\Delta x\cdot\beta\cdot\Delta x = 0$$

Divide by Δx & take $\lim_{\Delta x\to 0}$

$$\lim_{\Delta x\to 0} \frac{T+\Delta T}{\Delta x}\cdot\cos(\theta+\Delta\theta) - \frac{T}{\Delta x}\cos\theta = 0 \Rightarrow \boxed{\frac{d}{dx}\left(T\cdot\cos\theta\right) = 0} \quad (1)$$

$$\lim_{\Delta x\to 0} \frac{T+\Delta T}{\Delta x}\cdot\sin(\theta+\Delta\theta) - \frac{T}{\Delta x}\cdot\sin\theta - W(x) = 0 \quad \boxed{\frac{d}{dx}\left(T\cdot\sin\theta\right) - W(x) = 0} \quad (2)$$

$$\lim_{\Delta x\to 0} T\cdot\sin\theta - \frac{\Delta y}{\Delta x}\cdot T\cdot\cos\theta + W(x)\cdot\beta\cdot\Delta x = 0$$

$$\cancel{T}\cdot\sin\theta - \frac{dy}{dx}\cancel{T}\cdot\cos\theta = 0 \Rightarrow \boxed{\frac{dy}{dx} = \tan\theta} \quad (3)$$

Integrate (1) & (2):

(1) becomes : $T \cdot \cos\theta = \text{constant}$

$$\Rightarrow \boxed{T \cdot \cos\theta = F_H}$$

Horizontal component of tension

(2) $T \cdot \sin\theta = \int w(x) \cdot dx$

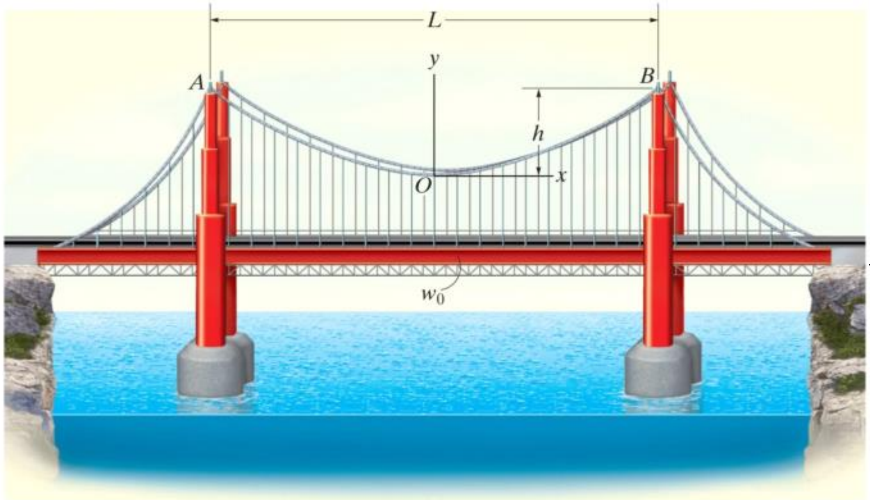
Divide (2)/(1): $\dfrac{\cancel{T}\sin\theta}{\cancel{T}\cos\theta} = \dfrac{1}{F_H}\int w(x) \cdot dx$

$$\tan\theta \rightarrow \frac{dy}{dx} = \frac{1}{F_H}\int w(x) \cdot dx$$

$$\boxed{y(x) = \frac{1}{F_H}\int\left(\int w(x) \cdot dx\right)dx}$$

Cable Curve Equation!

If F_H & $w(x)$ are known, integrate twice to get $y(x)$ with two integration constants, C_1 & C_2. Apply B.C.'s to find C_1 & C_2.



Determine the maximum force developed in the cable and the cable's required length. The span length and sag are known.

Where will the tension be lowest?

A) A   B) B

C) O   D) $x = L/4$

Will it be zero at the origin?

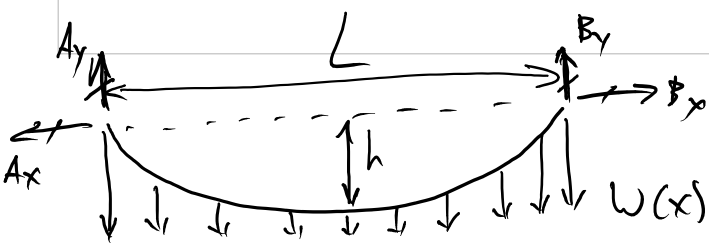
A) Yes

B) No

Tension at O is $T = F_H$

because $\theta = 0 \Rightarrow$

$F_H = T \cdot \cos\theta = T$



$w(x) = w_0$ (constant)

Cable equation: $$y(x) = \frac{1}{F_H} \cdot \int \left(\int w(x) \cdot dx \right) \cdot dx$$

$$= \frac{1}{F_H} \int \left(\int w_0 \cdot dx \right) dx$$

$$= \frac{w_0}{F_H} \int \left(\int dx \right) \cdot dx$$

$$= \frac{w_0}{F_H} \int \left(x + C_1 \right) \cdot dx$$

$$y(x) = \frac{w_0}{F_H} \left(\frac{x^2}{2} + C_1 \cdot x + C_2 \right)$$

# of unknowns?

A) 1

B) 2

C) 3 : C_1, C_2, F_H

D) 4

E) 0

3 Boundary conditions

$y(0)=0$

$y(\pm \frac{L}{2}) = h$

$\left.\frac{dy}{dx}\right|_{x=0} = 0$

$$y(0) = 0 = \frac{w_0}{F_H} \cdot \left[\cancel{\frac{(0)^2}{2}} + \cancel{C_1 \cdot (0)} + C_2 \right]$$

$$\Rightarrow C_2 = 0$$

$$y(x) = \frac{w_0}{F_H}\left(\frac{x^2}{2} + C_1 \cdot x\right)$$

$$\frac{dy}{dx} = \frac{w_0}{F_H}(x + C_1) \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = 0 = \frac{w_0}{F_H}(0 + C_1)$$

$$\Rightarrow C_1 = 0$$

$$y(x) = \frac{w_0 \cdot x^2}{2 \cdot F_H}$$

$$y(\pm \tfrac{L}{2}) = h = \frac{w_0 \cdot (\pm \frac{L}{2})^2}{2 \cdot F_H} \Rightarrow \boxed{F_H = \frac{w_0 \cdot L^2}{8h}}$$

$$\boxed{y(x) = \frac{\cancel{w_0} \cdot x^2}{\cancel{2}} \cdot \frac{\cancelto{4}{8}h}{\cancel{w_0} \cdot L^2} = 4 \cdot h \cdot \left(\frac{x}{L}\right)^2}$$

Parabolic, depends only on the span L and sag h.

$$\frac{dy}{dx} = \frac{8 \cdot h x}{L^2}$$

The tension can be found from

$$F_H = T \cdot \cos\theta \implies T = \frac{F_H}{\cos\theta}$$

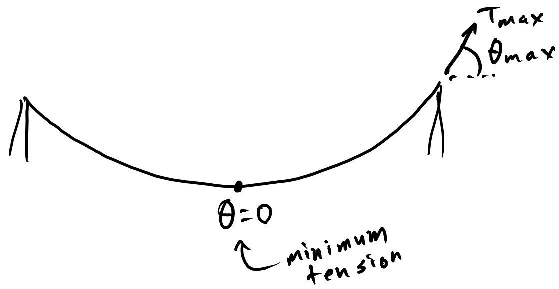
$$T = \frac{w_o L^2}{8 \cdot h \cdot \cos\theta}$$

Recall $\frac{dy}{dx} = \tan\theta \implies \theta = \tan^{-1}\left(\frac{dy}{dx}\right)$

$$\theta(x) = \tan^{-1}\left(\frac{8 \cdot h \cdot x}{L^2}\right)$$

... This result suggest that

$$\theta_{max} = \theta\left(x = \pm\frac{L}{2}\right) = \tan^{-1}\left(\frac{4 \cdot h}{L}\right)$$



$$\boxed{T_{max} = \frac{w_o \cdot L^2}{8 \cdot h \cdot \cos\left[\tan^{-1}\left(\frac{4h}{L}\right)\right]}}$$

Cable has a total length of the arclength of $y(x)$ from $-\frac{L}{2} < x < \frac{L}{2}$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

Total length $\mathcal{L} = \underbrace{\int ds}_{\text{length of cable}} = \int_{-L/2}^{L/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$

$$= \int_{-L/2}^{L/2} \sqrt{1 + \left(\frac{8hx}{L^2}\right)^2} \cdot dx$$

$$= 2 \int_0^{L/2} \sqrt{1 + \left(\frac{8hx}{L^2}\right)^2} \cdot dx$$

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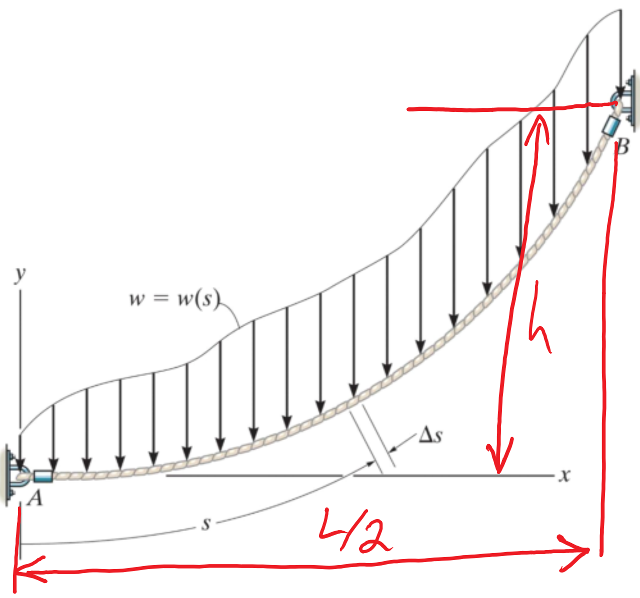
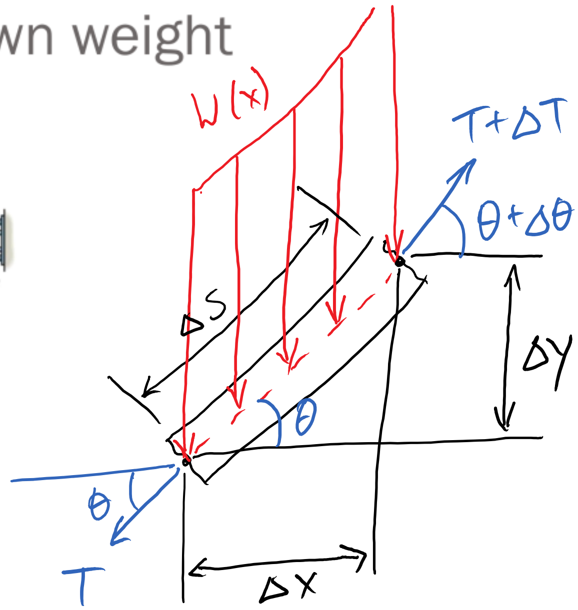
$$\boxed{\mathcal{L} = \frac{L}{2} \cdot \left[\sqrt{1 + \left(\frac{4h}{L}\right)^2} + \frac{L}{4 \cdot h} \cdot \sinh^{-1}\left(\frac{4 \cdot h}{L}\right) \right]}$$

Cable subjected to its own weight





We will use: $T\cdot\cos\theta = F_H$ (constant)

$$T\cdot\sin\theta = \int w(s)\cdot ds$$

$$\frac{dy}{dx} = \frac{1}{F_H}\cdot\int w(s)\cdot ds$$

$$ds^2 = dx^2 + dy^2$$

divide ds^2 by dx^2 to get

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \Rightarrow \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= \sqrt{1 + \left(\frac{1}{F_H}\int w(s)\cdot ds\right)^2}$$

$$\Rightarrow ds = \sqrt{1 + \left(\frac{1}{F_H}\int w(s)\,ds\right)^2}\,dx$$

$\Rightarrow$ we should be

$s \qquad$ fnc. of $s \qquad x$

$\Rightarrow$ we should be able to separate x & s variables and integrate

Rearrange:
$$dx = \frac{ds}{\sqrt{1+\left(\frac{1}{F_H}\int w(s)\cdot ds\right)^2}}$$

Integrate:
$$x = \int \frac{ds}{\sqrt{1+\left(\frac{1}{F_H}\int w(s)\cdot ds\right)^2}}$$

Now, $w(s) = w_0$ (constant in s).
$$x = \int \frac{ds}{\sqrt{1+\left(\frac{w_0}{F_H}\int ds\right)^2}} = \int \frac{ds}{\sqrt{1+\left(\frac{w_0\cdot s + C_1}{F_H}\right)^2}}$$

Let $u = \frac{w_0\cdot s + C_1}{F_H} \Rightarrow ds = \frac{F_H}{w_0}\cdot du$
$$x = \frac{F_H}{w_0}\int \frac{du}{\sqrt{1+u^2}} = \frac{F_H}{w_0}\left(\sinh^{-1}u + C_2\right)$$
$$x = \frac{F_H}{w_0}\left[\sinh^{-1}\left(\frac{w_0\cdot s + C_1}{F_H}\right) + C_2\right]$$

Apply boundary conditions: $y(0)=0$, $y\left(\frac{L}{2}\right)=h$
$$\left.\frac{dy}{dx}\right|_{x=0} = 0$$

Recall $\boxed{\frac{dy}{dx} = \frac{1}{F_H}\int w_0\cdot ds = \frac{1}{F_H}\cdot\left(w_0\cdot s + C_1\right)}$

$$\Rightarrow 0 = \frac{1}{F}\cdot\left(w_0\cdot 0 + C_1\right) \Rightarrow C_1 = 0$$

$$\Rightarrow \quad 0 = \frac{1}{F_H}\cdot\left(W_0 \cdot 0 + C_1\right) \Rightarrow C_1 = 0$$

$$y(x=0)=0 \Rightarrow 0 = \frac{F_H}{W_0}\left[\sinh^{-1}\left(\frac{W_0 \cdot 0}{F_H} + 0\right) + C_2\right]$$

$$= \frac{F_H}{W_0}\left[0 + C_2\right]$$

$$\Rightarrow C_2 = 0$$

$$x = \frac{F_H}{W_0}\cdot \sinh^{-1}\left(\frac{W_0 \cdot S}{F_H}\right)$$

$$\Rightarrow \sinh^{-1}\left(\frac{W_0 \cdot S}{F_H}\right) = \frac{W_0 \cdot x}{F_H} \Rightarrow \frac{W_0 \cdot S}{F_H} = \sinh\left(\frac{W_0 \cdot x}{F_H}\right)$$

See that $\frac{dy}{dx} = \frac{W_0 \cdot S}{F_H}$ $\Rightarrow \frac{dy}{dx} = \sinh\left(\frac{W_0 \cdot x}{F_H}\right)$

Integrate: $y(x) = \frac{F_H}{W_0}\left[\cosh\left(\frac{W_0 \cdot x}{F_H}\right)\right] + C_3$

Apply $y(0)=0$ $\Rightarrow 0 = \frac{F_H}{W_0}\left[\cosh(0)\right] + C_3$

$$\Rightarrow C_3 = -\frac{F_H}{W_0}$$

$$y(x) = \frac{F_H}{W_0}\cdot\left[\cosh\left(\frac{W_0 \cdot x}{F_H}\right) - 1\right]$$

Note: F_H is still an unknown.

Apply $y(x=\frac{L}{2})=h$ $\Rightarrow h = \frac{F_H}{W_0}\cdot\left[\cosh\left(\frac{W_0 \cdot L}{2 F_H}\right) - 1\right]$ ☆

The only way to solve for F_H is to substitute numerical values for h, W_0, & L and solve numerically.

One approach: Rearrange ⊛ as follows:

$$\underbrace{\frac{h \cdot w_0}{F_H} + 1}_{\text{call } z_1(F_H)} = \underbrace{\cosh\left(\frac{w_0 \cdot L}{2 \cdot F_H}\right)}_{\text{call } z_2(F_H)}$$

Plot $z_1(F_H) = \dfrac{h \cdot w_0 + F_H}{F_H}$ & $z_2(F_H) = \cosh\left(\dfrac{w_0 L}{2 F_H}\right)$

If $h \cdot w_0 = 1$ & $\dfrac{w_0 \cdot L}{2} = 1$ $\Rightarrow$

$$z_1(F_H) = \frac{1 + F_H}{F_H}$$

$$z_2(F_H) = \cosh\left(\frac{1}{F_H}\right)$$